# llective Instabilities in Wakefield Coupled Bunches

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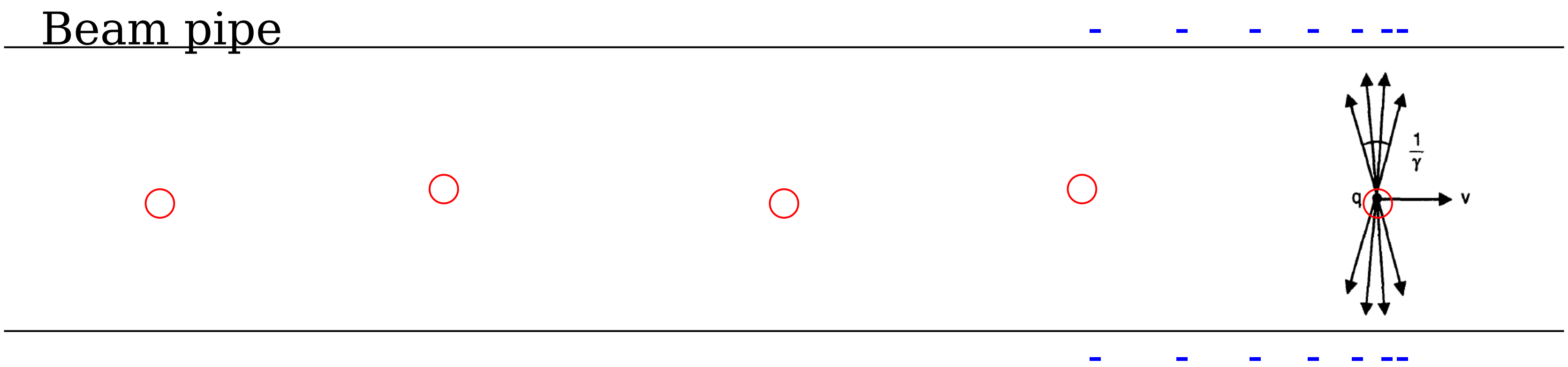
Liverpool Accelerator Group Meeting, Cockcroft 14 February 2007

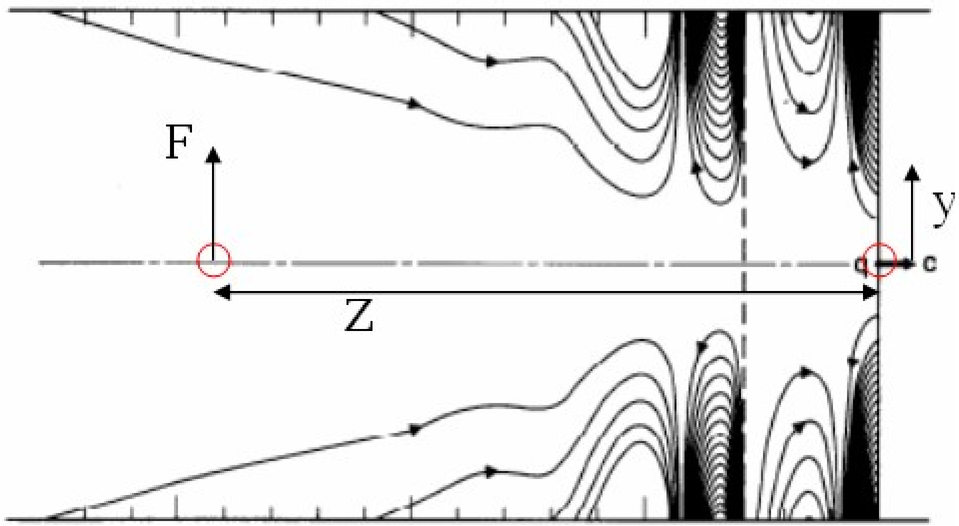



## Objective

- OCS6 Damping Ring
    - Transverse Growth Rates

Chao (1993)

# Uniform Resistive Wall



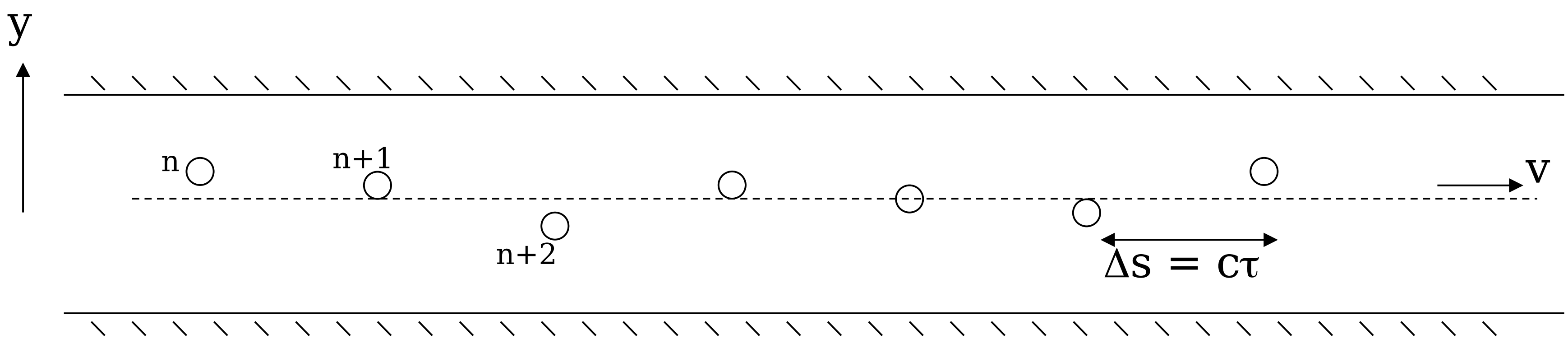

No wakefield this side

Transverse Force $F \propto W_1(z)\, y$

Wake potential $W_1(z) \propto \dfrac{1}{\sqrt{z}}$

Chao (1993)

# Equation of motion

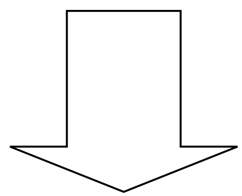



No wakefield

$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = 0$$

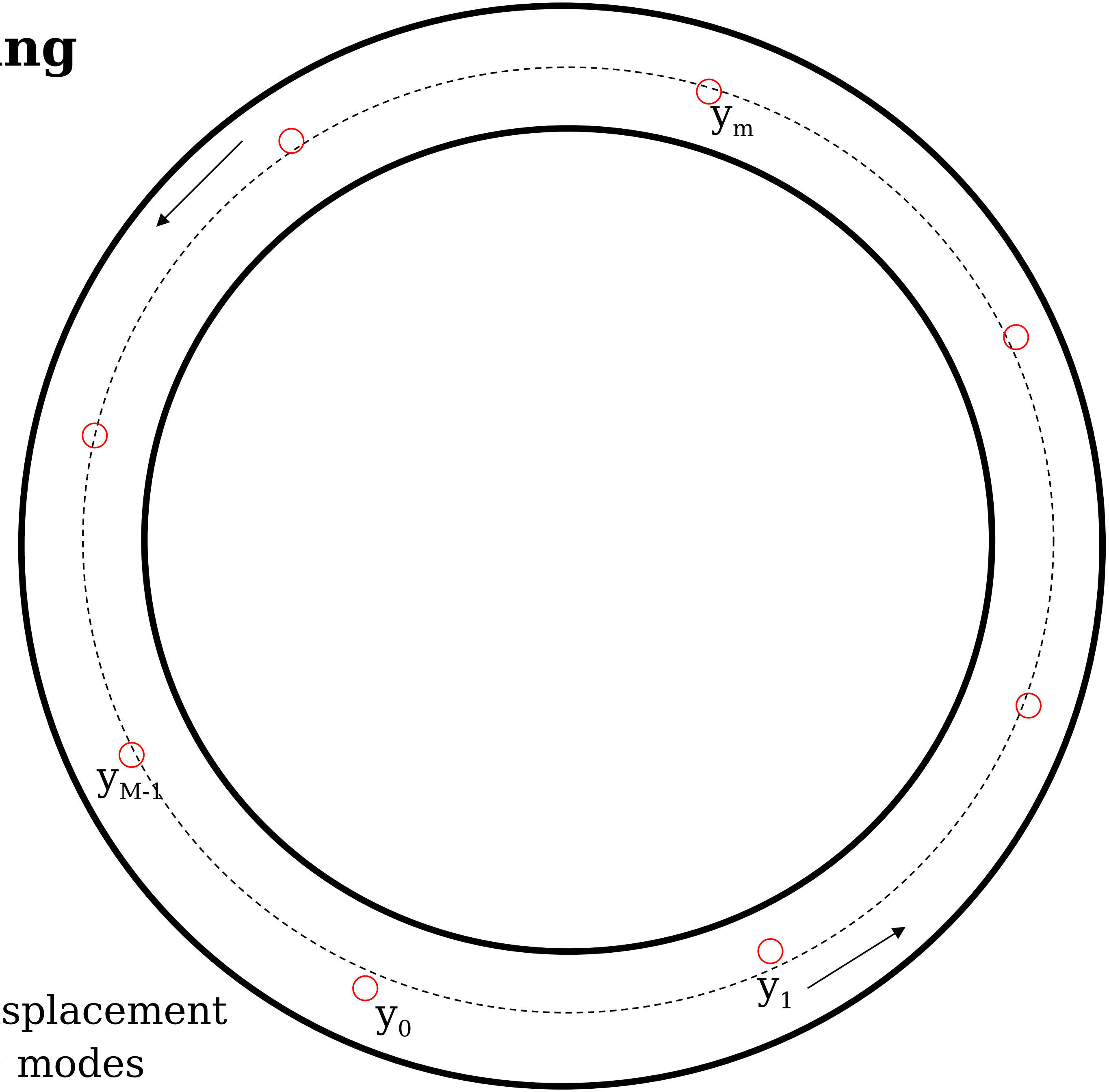
Wakefield from bunch ahead

$$-\frac{Nr_0 c}{\gamma T_0} W_1(-\Delta s) \quad y_{n+1}(t-\tau)$$



$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = -\frac{Nr_0 c}{\gamma T_0}\left[W_1(-\Delta s)\, y_{n+1}(t-\tau) + W_1(-2\Delta s)\, y_{n+2}(t-2\tau) + \ldots\right]$$

# Damping Ring




$y_n$ = transverse displacement
periodic nature → modes

# Modes

$$\ddot{y}_n(t) + \omega_\beta^2 y_n(t) = -\frac{Nr_0 c}{\gamma T_0}\left[W_1'(-\Delta s)\, y_{n+1}(t-\tau) + W_1'(-2\Delta s)\, y_{n+2}(t-2\tau) + \ldots\right]$$

Trial solution $$y_n(t) = y_n(0) e^{-i\Omega t}$$

Circulant Matrix
$$\begin{pmatrix} c_0(\lambda) & c_1(\lambda) & c_2(\lambda) & \ldots & c_{n-1}(\lambda) \\ c_{n-1}(\lambda) & c_0(\lambda) & c_0(\lambda) & \ldots & c_{n-2}(\lambda) \\ \ldots & & & & \ldots \\ \ldots & & & & \ldots \\ c_1(\lambda) & c_2(\lambda) & c_3(\lambda) & \ldots & c_0(\lambda) \end{pmatrix} y = \lambda y$$

Eigenvector / Mode $$\tilde{y}_\mu(t) = \sum_{m=0}^{M-1} y_m(t) e^{i\frac{2\pi m \mu}{N}}$$

(Gray 2006)

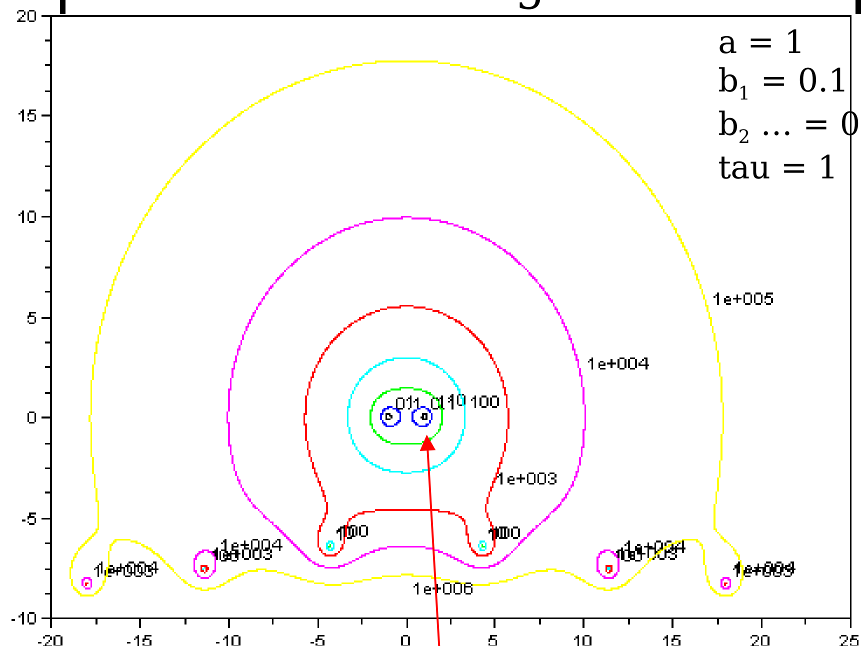
# Characteristic Equation

e.g. 2 bunches, Mode 0

$$-\Omega^2 + a = -b_1 e^{i\Omega\tau} - b_2 e^{i2\Omega\tau} - \dots - b_{2n-1} e^{i(2n-1)\Omega\tau} - b_{2n} e^{i2n\Omega\tau}$$

Multiple solutions:

|Left hand side – right hand side|




If assume dominated by betatron oscillation ...

## … derive analytic expression for small wakefield

$$\Omega_\mu - \omega_\beta = -\mathrm{i}\frac{MNr_0c}{2\gamma T_0^2\omega_\beta}\sum_{p=-\infty}^{\infty} Z_\perp\left[\omega_\beta + (pM+\mu)\omega_0\right]$$

Chao (1993)

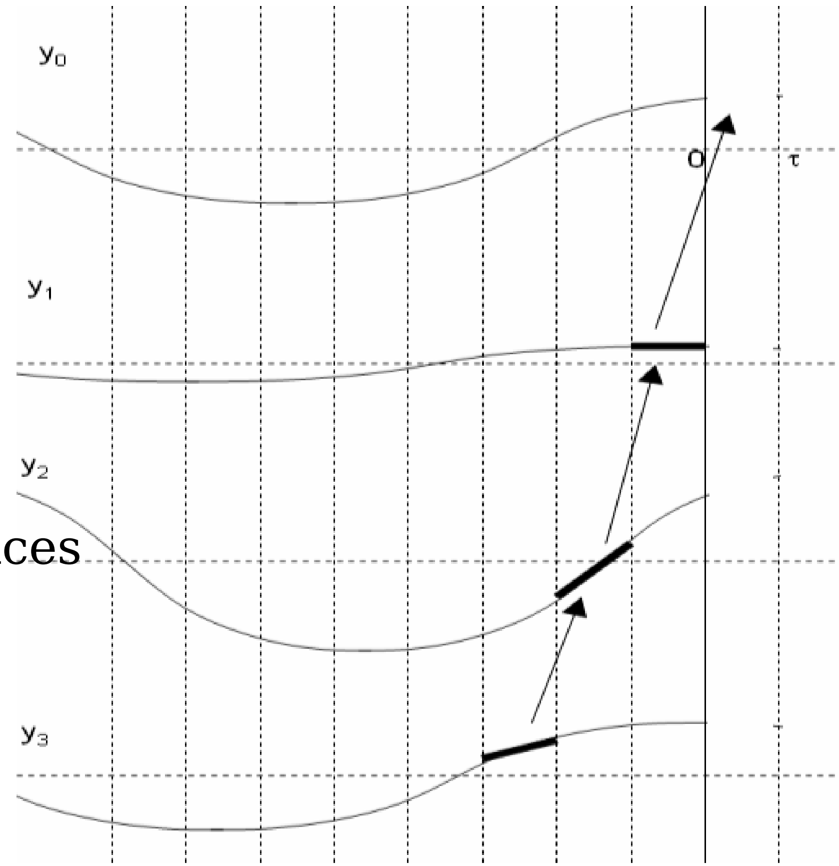
Mode $\tilde{y}_\mu(t) = \tilde{y}_\mu(0)e^{-i\Omega_\mu t}$

FFT $y_n(0)$

Growth Rate $\frac{1}{\tau^{(\mu)}} = \mathrm{Im}\Omega_\mu$

# Simulation Method

$$\ddot{y}_0(t) + \omega^2 y_0(t) = -w_1 y_1(t-\tau) - w_2 y_2(t-2\tau) - w_3 y_3(t-3\tau) - \ldots$$



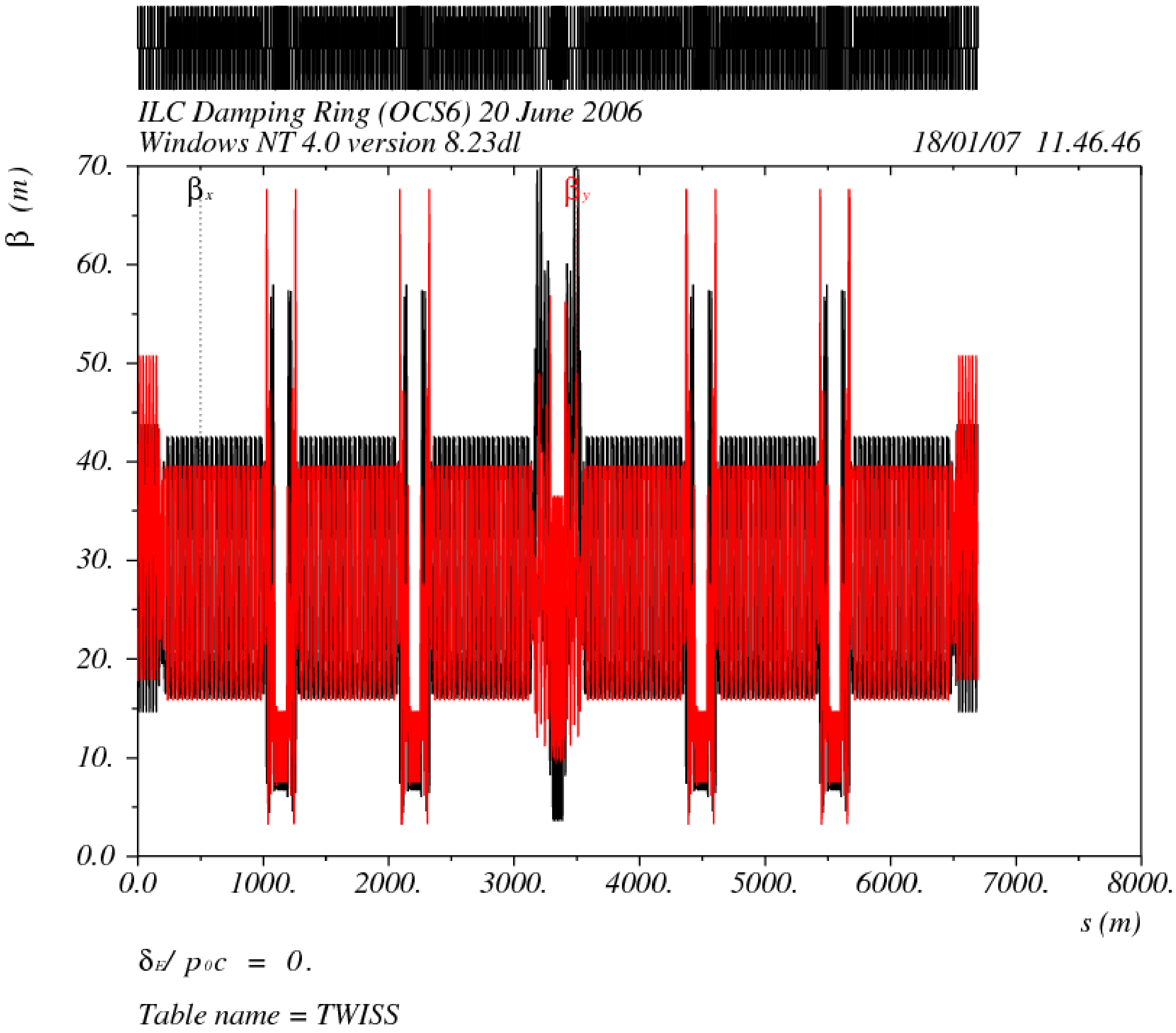

tegrate over one time interval between slices
epeat for next interval

$$y(\tau) \approx y_0 \cos\omega\tau + \frac{v_0}{\omega}\sin\omega\tau$$

$$\dot{y}(\tau) \approx -\omega y_0 \sin\omega\tau + v_0 \cos\omega\tau + q\tau$$

SHM Kick

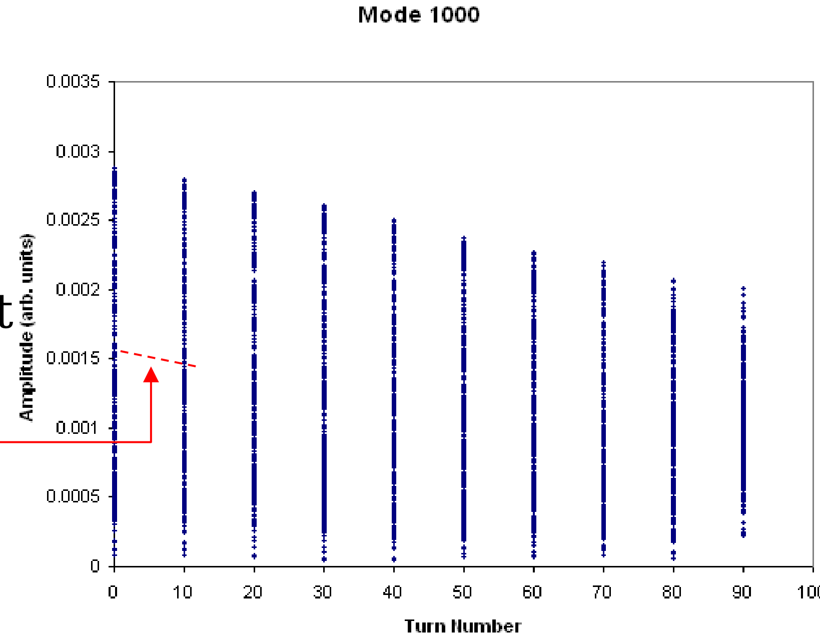
# OCS6 Damping Ring

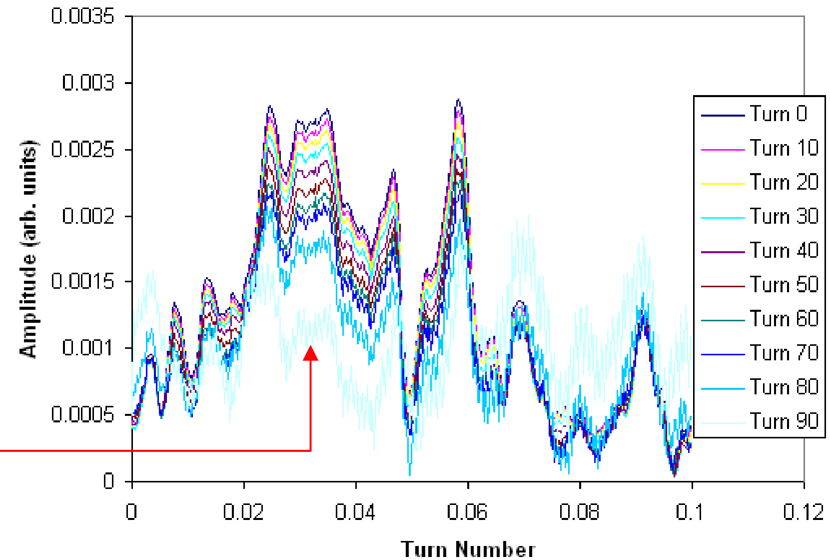


# Mode Amplitudes

- Mode amplitude ~ exp( t/τ)

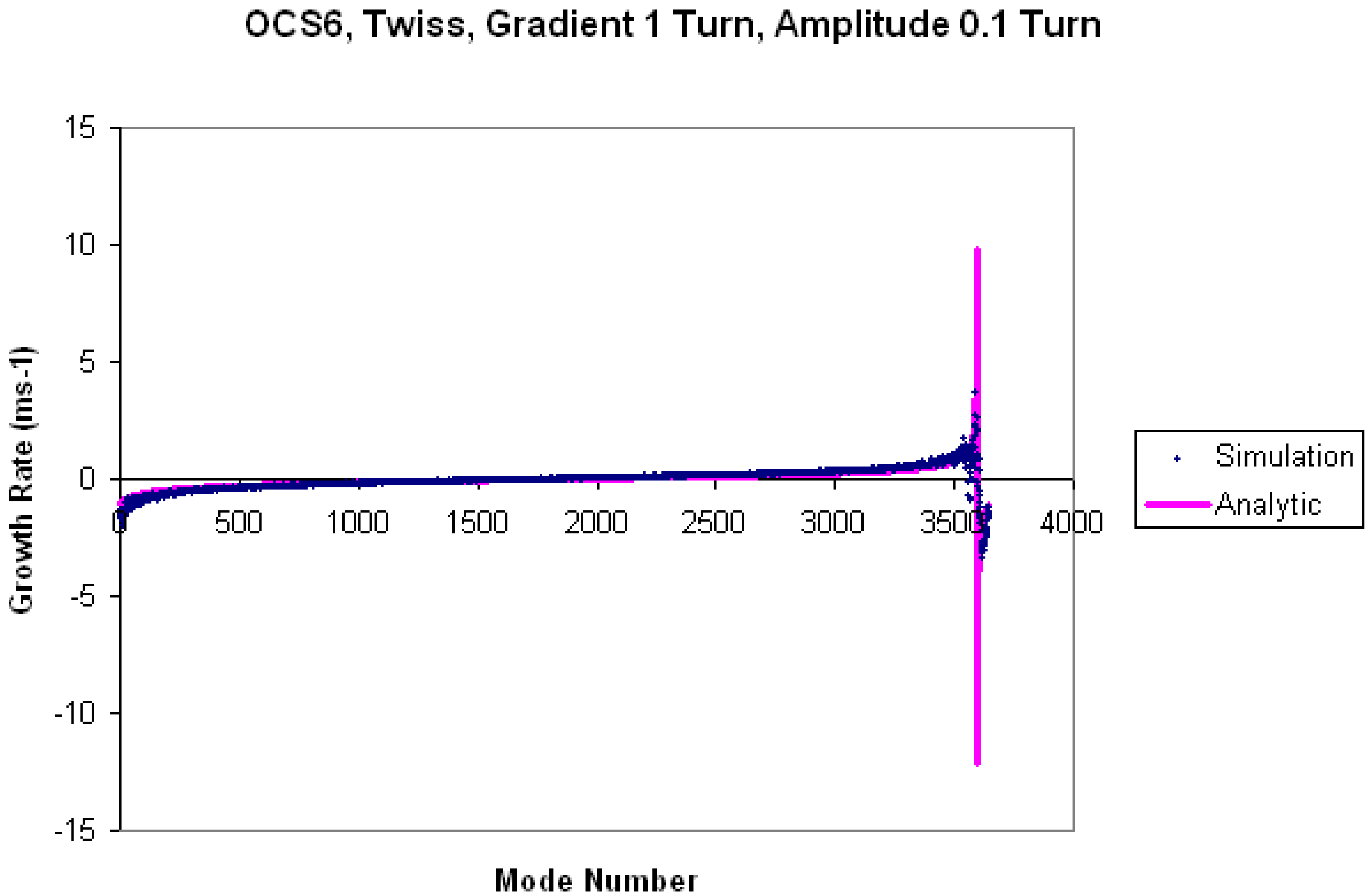
- Growth rate 1/τ ~ initial gradient




High frequency oscillations?

# OCS6 Growth Rate



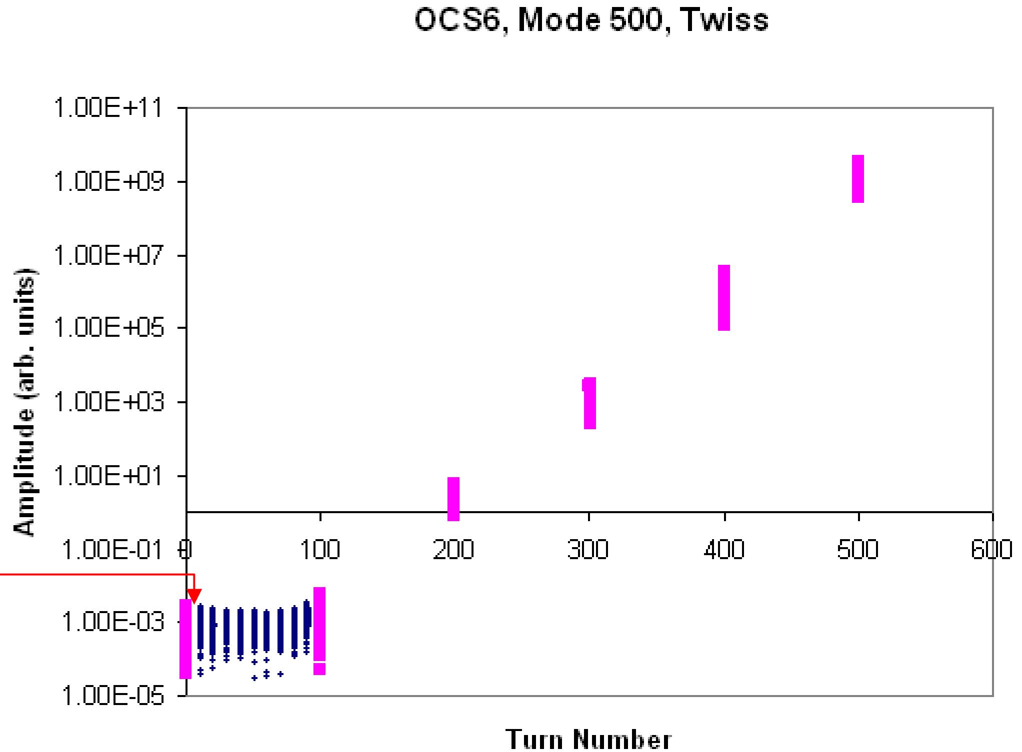

Assume constant beta for analytic curve.

# Problems

$e^{-i\Omega_\mu t}$ not complete. May also be $t^n e^{-i\Omega_\mu t}$ (Wright 1948)

$\tilde{y}_\mu(0)e^{-i\Omega_\mu t}$ not general. Could be $\tilde{y}_\mu(0)\left(c_1 e^{-i\Omega_\mu^{(1)} t} + c_2 e^{-i\Omega_\mu^{(2)} t} + c_3 e^{-i\Omega_\mu^{(3)} t} + \ldots\ldots\right)$

Non-exponential behaviour?




Growth rate ?

Thank You